

Lie Symmetries of Einstein's Vacuum Equations in N Dimensions

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Received January 20, 1997; Accepted October 14, 1997

Abstract

We investigate Lie symmetries of Einstein's vacuum equations in N dimensions, with a cosmological term. For this purpose, we first write down the second prolongation of the symmetry generating vector fields, and compute its action on Einstein's equations. Instead of setting to zero the coefficients of all independent partial derivatives (which involves a very complicated substitution of Einstein's equations), we set to zero the coefficients of derivatives that do not appear in Einstein's equations. This considerably constrains the coefficients of symmetry generating vector fields. Using the Lie algebra property of generators of symmetries and the fact that general coordinate transformations are symmetries of Einstein's equations, we are then able to obtain all the Lie symmetries. The method we have used can likely be applied to other types of equations.

1 Introduction

Consider a nondegenerate system of n -th order nonlinear partial differential equations for a number of independent variables x and dependent variables g :

$$\Delta_\nu \left(x, g, \partial g, \dots, \partial^{(n)} g \right) = 0. \quad (1)$$

Let v be a linear combination of first-order partial derivatives with respect to the x and g , with coefficients depending on the x and g . Then v will generate a Lie symmetry of Eq. (1) if and only if the following holds [1]:

$$\left[\text{pr}^{(n)} v \right] \Delta_\nu = 0 \quad \text{whenever} \quad \Delta_\nu = 0, \quad (2)$$

where $\text{pr}^{(n)} v$ is the so-called n -th prolongation of v . Eq. (2) constitutes a system of linear equations for the coefficients of partial derivatives making up the operator v .

To compute Eq. (2) explicitly, the main problem consists in eliminating nonindependent partial derivatives through substitution of $\Delta_\nu = 0$. This can be complicated, as illustrated by the case of the Yang-Mills equations examined elsewhere [2].

In this paper, we shall investigate Lie symmetries of Einstein's vacuum equations in N dimensions, including a cosmological term. Substitution of Eq. (1) is much more complicated here than in the Yang-Mills case. We will show, however, that the substitution can be bypassed by using the Lie algebra property of symmetry generators and knowledge of some of the symmetries. This technique can likely be used in other systems of nonlinear partial differential equations.

In Section 2, we write down Einstein's vacuum equations in N dimensions (Einstein's equations, for short), and recall some of their properties. In Section 3, we compute the action of the second prolongation of v on Einstein's equations. Coefficients of partial derivatives not appearing in Einstein's equations must vanish identically, and this is effected in Section 4. There result constraints on symmetry generators which, however, are not enough to determine the generators completely. In Section 5, we use the fact that general coordinate transformations are symmetries of Einstein's equations, together with the Lie algebra property of generators of symmetries, to show that the complete set of Lie symmetries of Einstein's equations coincides with general coordinate transformations and, when the cosmological term vanishes, uniform rescalings of the metric.

Lie symmetries [3] and generalized symmetries [4, 5] of the Einstein vacuum equations in 4 dimensions, without a cosmological term, were investigated before, with results in agreement with ours.

2 Einstein's vacuum equations in N dimensions

Einstein's vacuum equations in N dimensions can be written as [6, 7, 8]

$$R_{\alpha\beta} - \lambda g_{\alpha\beta} = 0. \quad (3)$$

Here λ is a constant, and $\lambda g_{\alpha\beta}$ is the cosmological term. $R_{\alpha\beta}$, the Ricci tensor, is given by

$$R_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} \{ -\partial_\gamma \partial_\delta g_{\alpha\beta} - \partial_\alpha \partial_\beta g_{\gamma\delta} + \partial_\beta \partial_\delta g_{\alpha\gamma} + \partial_\alpha \partial_\gamma g_{\delta\beta} \} + g^{\gamma\delta} g^{\tau\rho} \{ \Gamma_{\tau\gamma\alpha} \Gamma_{\rho\delta\beta} - \Gamma_{\tau\gamma\delta} \Gamma_{\rho\alpha\beta} \}. \quad (4)$$

The symbol ∂_γ represents a partial derivative with respect to the independent variable x^γ ($\gamma = 1, \dots, N$). The $g_{\nu\lambda} = g_{\lambda\nu}$ are dependent variables. The $g^{\mu\nu}$ are defined so that

$$g^{\mu\nu} g_{\nu\lambda} = \delta_\mu^\lambda, \quad (5)$$

where δ_μ^λ is the Kronecker delta. The Christoffel symbols $\Gamma_{\tau\gamma\alpha}$ are given by

$$\Gamma_{\tau\gamma\alpha} = \frac{1}{2} \{ \partial_\alpha g_{\tau\gamma} + \partial_\gamma g_{\tau\alpha} - \partial_\tau g_{\gamma\alpha} \}. \quad (6)$$

Note that we have

$$\partial_\alpha g_{\tau\gamma} = \Gamma_{\tau\gamma\alpha} + \Gamma_{\gamma\tau\alpha}. \quad (7)$$

There are $N(N+1)/2$ variables $g_{\nu\lambda}$. Varying them independently, we get from Eq. (5)

$$\delta g^{\mu\nu} = -g^{\mu\kappa}(\delta g_{\kappa\lambda})g^{\nu\lambda}, \quad (8)$$

whence

$$-\frac{\partial g^{\mu\nu}}{\partial g_{\kappa\lambda}} = X^{\mu\nu\kappa\lambda} \equiv \begin{cases} g^{\mu\kappa}g^{\nu\lambda} + g^{\mu\lambda}g^{\nu\kappa} & \text{if } \kappa \neq \lambda, \\ g^{\mu\kappa}g^{\nu\lambda} & \text{if } \kappa = \lambda. \end{cases} \quad (9)$$

In the symbol $X^{\mu\nu\kappa\lambda}$, indices can be lowered, for instance

$$X_{\mu\nu}{}^{\kappa\lambda} = \begin{cases} \delta_\mu^\kappa\delta_\nu^\lambda + \delta_\mu^\lambda\delta_\nu^\kappa & \text{if } \kappa \neq \lambda, \\ \delta_\mu^\kappa\delta_\nu^\lambda & \text{if } \kappa = \lambda. \end{cases} \quad (10)$$

Note that we have

$$\frac{\partial g_{\mu\nu}}{\partial g_{\kappa\lambda}} = X_{\mu\nu}{}^{\kappa\lambda}. \quad (11)$$

For later purposes, we now evaluate the partial derivatives of the Ricci tensor with respect to the metric tensor and its partial derivatives. For this, the X symbol is particularly useful. From Eqs. (4), (9) and (11), we find that

$$\frac{\partial R_{\alpha\beta}}{\partial(\partial_\kappa\partial_\lambda g_{\mu\nu})} = \frac{1}{2}g^{\gamma\delta}\left\{-X_{\gamma\delta}{}^{\kappa\lambda}X_{\alpha\beta}{}^{\mu\nu} - X_{\alpha\beta}{}^{\kappa\lambda}X_{\gamma\delta}{}^{\mu\nu} + X_{\delta\beta}{}^{\kappa\lambda}X_{\gamma\alpha}{}^{\mu\nu} + X_{\gamma\alpha}{}^{\kappa\lambda}X_{\delta\beta}{}^{\mu\nu}\right\}, \quad (12)$$

$$\begin{aligned} \frac{\partial R_{\alpha\beta}}{\partial(\partial_\kappa g_{\mu\nu})} = & \frac{1}{2}g^{\gamma\delta}g^{\tau\rho}\left\{\left[\delta_\alpha^\kappa X_{\tau\gamma}{}^{\mu\nu} + \delta_\gamma^\kappa X_{\tau\alpha}{}^{\mu\nu} - \delta_\tau^\kappa X_{\gamma\alpha}{}^{\mu\nu}\right]\Gamma_{\rho\delta\beta} \right. \\ & + \left[\delta_\beta^\kappa X_{\rho\delta}{}^{\mu\nu} + \delta_\delta^\kappa X_{\rho\beta}{}^{\mu\nu} - \delta_\rho^\kappa X_{\delta\beta}{}^{\mu\nu}\right]\Gamma_{\tau\gamma\alpha} \\ & - \left[\delta_\delta^\kappa X_{\tau\gamma}{}^{\mu\nu} + \delta_\gamma^\kappa X_{\tau\delta}{}^{\mu\nu} - \delta_\tau^\kappa X_{\gamma\delta}{}^{\mu\nu}\right]\Gamma_{\rho\alpha\beta} \\ & \left. - \left[\delta_\beta^\kappa X_{\rho\alpha}{}^{\mu\nu} + \delta_\alpha^\kappa X_{\rho\beta}{}^{\mu\nu} - \delta_\rho^\kappa X_{\alpha\beta}{}^{\mu\nu}\right]\Gamma_{\tau\gamma\delta}\right\}, \end{aligned} \quad (13)$$

$$\begin{aligned} \frac{\partial R_{\alpha\beta}}{\partial g_{\mu\nu}} = & \frac{1}{2}\{\partial_\gamma\partial_\delta g_{\alpha\beta} + \partial_\alpha\partial_\beta g_{\gamma\delta} - \partial_\delta\partial_\beta g_{\gamma\alpha} - \partial_\gamma\partial_\alpha g_{\delta\beta}\}X^{\gamma\delta\mu\nu} \\ & - \{\Gamma_{\tau\gamma\alpha}\Gamma_{\rho\delta\beta} - \Gamma_{\tau\gamma\delta}\Gamma_{\rho\alpha\beta}\}\left\{g^{\gamma\delta}X^{\tau\rho\mu\nu} + g^{\tau\rho}X^{\gamma\delta\mu\nu}\right\}. \end{aligned} \quad (14)$$

A word on notations. The summation convention on repeated indices has hitherto been used. There will, however, be instances where we will not want to use it. Following [2], we will put carets on indices wherever summation should not be carried out. This means, for instance, that in an equation like

$$M_\mu^\mu = N_{\hat{\alpha}}^{\hat{\alpha}}, \quad (15)$$

summation is carried out over μ but not over $\hat{\alpha}$, the latter index having a specific value. Furthermore, when dealing with symmetric matrices we will often need to restrict a summation to distinct values of a pair of indices. In that case, parentheses will enclose the indices, for instance

$$A_{(\mu\nu)}B^{\mu\nu} \equiv \sum_{\mu \leq \nu} A_{\mu\nu}B^{\mu\nu}. \quad (16)$$

Note that we have, for any symmetric A

$$A_{(\mu\nu)} X_{\gamma\delta}^{\mu\nu} = A_{\gamma\delta}. \quad (17)$$

In closing this section, we should point out that not all second-order partial derivatives of the metric tensor appear in the Ricci tensor. Indeed writing down the second-order derivatives explicitly, one can see that for any values of ρ and $\hat{\sigma}$, no terms like $\partial_\rho \partial_{\hat{\sigma}} g_{\hat{\sigma}\hat{\sigma}}$ or $\partial_{\hat{\sigma}} \partial_{\hat{\sigma}} g_{\rho\hat{\sigma}}$ appear in Eq. (4).

3 Prolongation of vector fields

The generator of a Lie symmetry of Einstein's vacuum equations in N dimensions has the form

$$v = H^\mu \frac{\partial}{\partial x^\mu} + \Phi_{(\mu\nu)} \frac{\partial}{\partial g_{\mu\nu}}. \quad (18)$$

Here H^μ and $\Phi_{\mu\nu}$ are functions of the independent variables x^λ and dependent variables $g_{\alpha\beta}$. The second summation on the right-hand side is restricted so that dependent variables are not counted twice. Nevertheless, it is useful to define $\Phi_{\mu\nu}$ for $\mu > \nu$ also, by setting $\Phi_{\mu\nu} = \Phi_{\nu\mu}$.

The second prolongation of v is given by [1]

$$\text{pr}^{(2)}v = H^\mu \frac{\partial}{\partial x^\mu} + \Phi_{(\mu\nu)} \frac{\partial}{\partial g_{\mu\nu}} + \Phi_{(\mu\nu)\kappa} \frac{\partial}{\partial(\partial_\kappa g_{\mu\nu})} + \Phi_{(\mu\nu)(\kappa\lambda)} \frac{\partial}{\partial(\partial_\kappa \partial_\lambda g_{\mu\nu})}. \quad (19)$$

Here also, summations are restricted so that identical objects are not counted twice. The $\Phi_{\mu\nu\kappa}$ and $\Phi_{\mu\nu\kappa\lambda}$ are functions of the independent and dependent variables that will soon be examined. Again, it is useful to extend the range of indices so that

$$\Phi_{\mu\nu\kappa} = \Phi_{\nu\mu\kappa} \quad \text{and} \quad \Phi_{\mu\nu\kappa\lambda} = \Phi_{\mu\nu\lambda\kappa} = \Phi_{\nu\mu\kappa\lambda}. \quad (20)$$

We now apply the right-hand side of Eq. (19) on the left-hand side of Eq. (3), and substitute Eqs. (11), (12), (13) and (14). Making use of Eq. (17) and rearranging, we find that all restrictions on summations disappear, and we obtain

$$\begin{aligned} [\text{pr}^{(2)}v] \{R_{\alpha\beta} - \lambda g_{\alpha\beta}\} &= -\lambda \Phi_{\alpha\beta} + \frac{1}{2} \Phi^{\gamma\delta} \{\partial_\gamma \partial_\delta g_{\alpha\beta} + \partial_\alpha \partial_\beta g_{\gamma\delta} - \partial_\delta \partial_\beta g_{\gamma\alpha} - \partial_\gamma \partial_\alpha g_{\delta\beta}\} \\ &\quad - \{\Gamma_{\tau\gamma\alpha} \Gamma_{\rho\delta\beta} - \Gamma_{\tau\gamma\delta} \Gamma_{\rho\alpha\beta}\} \left\{ g^{\gamma\delta} \Phi^{\tau\rho} + g^{\tau\rho} \Phi^{\gamma\delta} \right\} \\ &\quad + \frac{1}{2} g^{\gamma\delta} g^{\tau\rho} \{[\Phi_{\tau\gamma\alpha} + \Phi_{\tau\alpha\gamma} - \Phi_{\gamma\alpha\tau}] \Gamma_{\rho\delta\beta} + [\Phi_{\rho\delta\beta} + \Phi_{\rho\beta\delta} - \Phi_{\delta\beta\rho}] \Gamma_{\tau\gamma\alpha} \} \\ &\quad - [\Phi_{\tau\gamma\delta} + \Phi_{\tau\delta\gamma} - \Phi_{\gamma\delta\tau}] \Gamma_{\rho\alpha\beta} - [\Phi_{\rho\alpha\beta} + \Phi_{\rho\beta\alpha} - \Phi_{\alpha\beta\rho}] \Gamma_{\tau\gamma\delta} \} \\ &\quad + \frac{1}{2} g^{\gamma\delta} \{-\Phi_{\alpha\beta\gamma\delta} - \Phi_{\gamma\delta\alpha\beta} + \Phi_{\gamma\alpha\delta\beta} + \Phi_{\delta\beta\gamma\alpha}\}. \end{aligned} \quad (21)$$

The functions $\Phi_{\tau\gamma\alpha}$ and $\Phi_{\alpha\beta\gamma\delta}$ are given by [1]

$$\Phi_{\tau\gamma\alpha} = D_\alpha \{\Phi_{\tau\gamma} - H^\eta \partial_\eta g_{\tau\gamma}\} + H^\eta \partial_\alpha \partial_\eta g_{\tau\gamma} \quad (22)$$

and

$$\Phi_{\alpha\beta\gamma\delta} = D_\gamma D_\delta \{ \Phi_{\alpha\beta} - H^\eta \partial_\eta g_{\alpha\beta} \} + H^\eta \partial_\gamma \partial_\delta \partial_\eta g_{\alpha\beta}. \quad (23)$$

Here D_α is the total derivative operator, given by

$$D_\alpha = \partial_\alpha + \partial_\alpha g_{(\mu\nu)} \frac{\partial}{\partial g_{\mu\nu}} + \partial_\alpha \partial_\kappa g_{(\mu\nu)} \frac{\partial}{\partial(\partial_\kappa g_{\mu\nu})} + \partial_\alpha \partial_{(\kappa} \partial_{\lambda)} g_{(\mu\nu)} \frac{\partial}{\partial(\partial_\kappa \partial_\lambda g_{\mu\nu})}. \quad (24)$$

Substituting Eq. (24) in (22) and (23), we obtain

$$\Phi_{\tau\gamma\alpha} = \partial_\alpha \Phi_{\tau\gamma} - [\partial_\eta g_{\tau\gamma}] \partial_\alpha H^\eta + \partial_\alpha g_{(\mu\nu)} \frac{\partial \Phi_{\tau\gamma}}{\partial g_{\mu\nu}} - [\partial_\alpha g_{(\mu\nu)}] \partial_\eta g_{\tau\gamma} \frac{\partial H^\eta}{\partial g_{\mu\nu}} \quad (25)$$

and

$$\begin{aligned} \Phi_{\alpha\beta\gamma\delta} = & \partial_\gamma \partial_\delta \Phi_{\alpha\beta} - [\partial_\eta g_{\alpha\beta}] \partial_\gamma \partial_\delta H^\eta + \partial_\delta g_{(\mu\nu)} \partial_\gamma \left(\frac{\partial \Phi_{\alpha\beta}}{\partial g_{\mu\nu}} \right) + \partial_\gamma g_{(\mu\nu)} \partial_\delta \left(\frac{\partial \Phi_{\alpha\beta}}{\partial g_{\mu\nu}} \right) \\ & - [\partial_\gamma g_{(\mu\nu)}] \partial_\eta g_{\alpha\beta} \partial_\delta \left(\frac{\partial H^\eta}{\partial g_{\mu\nu}} \right) - [\partial_\delta g_{(\mu\nu)}] \partial_\eta g_{\alpha\beta} \partial_\gamma \left(\frac{\partial H^\eta}{\partial g_{\mu\nu}} \right) \\ & + [\partial_\gamma g_{(\mu\nu)}] \partial_\delta g_{(\pi\sigma)} \frac{\partial^2 \Phi_{\alpha\beta}}{\partial g_{\mu\nu} \partial g_{\pi\sigma}} - [\partial_\gamma g_{(\mu\nu)}] [\partial_\delta g_{(\pi\sigma)}] \partial_\eta g_{\alpha\beta} \frac{\partial^2 H^\eta}{\partial g_{\mu\nu} \partial g_{\pi\sigma}} \\ & - [\partial_\delta \partial_\eta g_{\alpha\beta}] \partial_\gamma H^\eta - [\partial_\gamma \partial_\eta g_{\alpha\beta}] \partial_\delta H^\eta + \partial_\gamma \partial_\delta g_{(\mu\nu)} \frac{\partial \Phi_{\alpha\beta}}{\partial g_{\mu\nu}} \\ & - [\partial_\gamma g_{(\mu\nu)}] \partial_\delta \partial_\eta g_{\alpha\beta} \frac{\partial H^\eta}{\partial g_{\mu\nu}} - [\partial_\delta g_{(\mu\nu)}] \partial_\gamma \partial_\eta g_{\alpha\beta} \frac{\partial H^\eta}{\partial g_{\mu\nu}} - [\partial_\eta g_{\alpha\beta}] \partial_\delta \partial_\gamma g_{(\mu\nu)} \frac{\partial H^\eta}{\partial g_{\mu\nu}}. \end{aligned} \quad (26)$$

Note that Eqs. (25) and (26) are consistent with (20).

The action of the second prolongation of v on the left-hand side of Eq. (3) can now be obtained by substituting (25) and (26) into (21). The resulting equation is very complicated but, fortunately, we will not have to write it down all at once. In any case, the conditions under which it vanishes, subject to Eq. (3), must be found so as to determine the Lie symmetries of (3).

4 Determining equations

In this section, we will consider in turn several combinations of partial derivatives of $g_{\mu\nu}$ appearing in Eq. (21).

$\partial g \partial \partial g$ terms

There are three groups of $\partial g \partial \partial g$ terms in Eq. (26). When they are substituted in (21), that makes altogether twelve groups of terms given by

$$\begin{aligned} \partial g \partial \partial g \rightarrow & \frac{1}{2} g^{\gamma\delta} \frac{\partial H^\eta}{\partial g_{(\mu\nu)}} \{ \partial_\gamma g_{\mu\nu} \partial_\delta \partial_\eta g_{\alpha\beta} + \partial_\delta g_{\mu\nu} \partial_\gamma \partial_\eta g_{\alpha\beta} + \partial_\eta g_{\alpha\beta} \partial_\delta \partial_\gamma g_{\mu\nu} \\ & + \partial_\alpha g_{\mu\nu} \partial_\beta \partial_\eta g_{\gamma\delta} + \partial_\beta g_{\mu\nu} \partial_\alpha \partial_\eta g_{\gamma\delta} + \partial_\eta g_{\gamma\delta} \partial_\alpha \partial_\beta g_{\mu\nu} \\ & - \partial_\delta g_{\mu\nu} \partial_\beta \partial_\eta g_{\alpha\gamma} - \partial_\beta g_{\mu\nu} \partial_\delta \partial_\eta g_{\alpha\gamma} - \partial_\eta g_{\alpha\gamma} \partial_\delta \partial_\beta g_{\mu\nu} \\ & - \partial_\gamma g_{\mu\nu} \partial_\alpha \partial_\eta g_{\delta\beta} - \partial_\alpha g_{\mu\nu} \partial_\gamma \partial_\eta g_{\delta\beta} - \partial_\eta g_{\delta\beta} \partial_\alpha \partial_\gamma g_{\mu\nu} \}. \end{aligned} \quad (27)$$

Rearranging indices, one can show that Eq. (27) becomes

$$\begin{aligned} \partial g \partial \partial g \rightarrow & \frac{1}{2} \partial_\gamma g_{(\mu\nu)} \partial_\delta \partial_\eta g_{(\rho\sigma)} \left\{ 2g^{\gamma\delta} X_{\alpha\beta}^{\rho\sigma} \frac{\partial H^\eta}{\partial g_{\mu\nu}} + g^{\eta\delta} X_{\alpha\beta}^{\mu\nu} \frac{\partial H^\gamma}{\partial g_{\rho\sigma}} \right. \\ & + (\delta_\alpha^\gamma \delta_\beta^\delta + \delta_\alpha^\delta \delta_\beta^\gamma) G^{\rho\sigma} \frac{\partial H^\eta}{\partial g_{\mu\nu}} + \delta_\alpha^\delta \delta_\beta^\eta G^{\mu\nu} \frac{\partial H^\gamma}{\partial g_{\rho\sigma}} \\ & - \delta_\beta^\delta X_\alpha^{\gamma\rho\sigma} \frac{\partial H^\eta}{\partial g_{\mu\nu}} - \delta_\beta^\gamma X_\alpha^{\delta\rho\sigma} \frac{\partial H^\eta}{\partial g_{\mu\nu}} - \delta_\beta^\eta X_\alpha^{\delta\mu\nu} \frac{\partial H^\gamma}{\partial g_{\rho\sigma}} \\ & \left. - \delta_\alpha^\delta X_\beta^{\gamma\sigma\rho} \frac{\partial H^\eta}{\partial g_{\mu\nu}} - \delta_\alpha^\gamma X_\beta^{\delta\sigma\rho} \frac{\partial H^\eta}{\partial g_{\mu\nu}} - \delta_\alpha^\eta X_\beta^{\delta\nu\mu} \frac{\partial H^\gamma}{\partial g_{\rho\sigma}} \right\}, \end{aligned} \quad (28)$$

where

$$G^{\rho\sigma} = \begin{cases} g^{\rho\sigma} & \text{if } \rho = \sigma, \\ 2g^{\rho\sigma} & \text{if } \rho \neq \sigma. \end{cases} \quad (29)$$

There are no $\partial_\sigma \partial_{\hat{\sigma}} g_{\rho\hat{\sigma}}$ terms in the Einstein equations, and no first-degree $\partial_\gamma g_{\mu\nu}$ terms at all. Derivatives like $\partial_\gamma g_{\mu\nu} \partial_{\hat{\sigma}} \partial_{\hat{\sigma}} g_{\rho\hat{\sigma}}$, for $\mu \leq \nu$ and $\rho \leq \hat{\sigma}$, are therefore independent. Setting the corresponding coefficients in Eq. (28) to zero yields $\forall \alpha, \beta, \gamma, (\mu\nu), (\rho\hat{\sigma})$

$$\begin{aligned} 0 = & \left\{ 2g^{\gamma\hat{\sigma}} X_{\alpha\beta}^{\rho\hat{\sigma}} + (\delta_\alpha^\gamma \delta_\beta^\hat{\sigma} + \delta_\alpha^\hat{\sigma} \delta_\beta^\gamma) G^{\rho\hat{\sigma}} - \delta_\beta^\hat{\sigma} X_\alpha^{\gamma\rho\hat{\sigma}} - \delta_\beta^\gamma X_\alpha^{\hat{\sigma}\rho\hat{\sigma}} - \delta_\alpha^\hat{\sigma} X_\beta^{\gamma\hat{\sigma}\rho} \right. \\ & \left. - \delta_\alpha^\gamma X_\beta^{\hat{\sigma}\hat{\sigma}\rho} \right\} \frac{\partial H^\hat{\sigma}}{\partial g_{\mu\nu}} + \left\{ g^{\hat{\sigma}\hat{\sigma}} X_{\alpha\beta}^{\mu\nu} + \delta_\alpha^\hat{\sigma} \delta_\beta^\hat{\sigma} G^{\mu\nu} - \delta_\beta^\hat{\sigma} X_\alpha^{\hat{\sigma}\mu\nu} - \delta_\alpha^\hat{\sigma} X_\beta^{\hat{\sigma}\nu\mu} \right\} \frac{\partial H^\gamma}{\partial g_{\rho\hat{\sigma}}}. \end{aligned} \quad (30)$$

We set $\alpha \neq \hat{\sigma}$, $\alpha \neq \gamma$, $\beta \neq \hat{\sigma}$, and $\beta \neq \gamma$. In three or more dimensions, this yields $\forall \gamma, (\mu\nu), (\rho\hat{\sigma})$

$$0 = g^{\hat{\sigma}\hat{\sigma}} X_{\alpha\beta}^{\mu\nu} \frac{\partial H^\gamma}{\partial g_{\rho\hat{\sigma}}}. \quad (31)$$

Since this must hold as an identity, we conclude that $\forall \gamma, (\rho\hat{\sigma})$

$$0 = \frac{\partial H^\gamma}{\partial g_{\rho\hat{\sigma}}}. \quad (32)$$

That is, all partial derivatives of H^γ with respect to components of the metric tensor vanish.

In two dimensions, the restrictions on indices before Eq. (31) imply that $\alpha = \beta \neq \gamma = \hat{\sigma}$. From this we conclude that $\forall (\rho\hat{\sigma})$

$$0 = \frac{\partial H^\hat{\sigma}}{\partial g_{\rho\hat{\sigma}}}. \quad (33)$$

Now set $\alpha = \beta \neq \rho = \hat{\sigma}$ in Eq. (30). We find $\forall \gamma, (\mu\nu), \hat{\alpha} \neq \hat{\sigma}$

$$0 = g^{\hat{\sigma}\hat{\sigma}} X_{\hat{\alpha}\hat{\alpha}}^{\mu\nu} \frac{\partial H^\gamma}{\partial g_{\hat{\sigma}\hat{\sigma}}}, \quad (34)$$

from which we conclude that $\forall \gamma, \hat{\sigma}$

$$0 = \frac{\partial H^\gamma}{\partial g_{\hat{\sigma}\hat{\sigma}}}. \quad (35)$$

Eqs. (33) and (35) cover all partial derivatives except $\partial H^1 / \partial g_{12}$. But that is easily seen to vanish by setting $\gamma = \rho = 1$, $\hat{\sigma} = 2$, and $\alpha = \beta = \mu = \nu = 1$ in (30). Therefore, Eq. (32) also holds in two dimensions.

From (27), one easily sees that (32) is sufficient for all $\partial g \partial \partial g$ terms to vanish.

$\partial g \partial g \partial g$ terms

$\partial g \partial g \partial g$ terms come from the substitution of Eqs. (25) and (26) in (21). Since they are always multiplied by first or second derivatives of H^η with respect to $g_{\mu\nu}$, they vanish identically due to (32).

$\partial \partial g$ terms

There are explicit $\partial \partial g$ terms in Eq. (21), and implicit ones through Eq. (26). Regrouping all those terms and rearranging indices, we find that they are given by

$$\begin{aligned} \partial \partial g \rightarrow & \frac{1}{2} \partial_\delta \partial_\eta g_{\rho\sigma} \left\{ \delta_\alpha^\rho \delta_\beta^\sigma \Phi^{\delta\eta} + \delta_\alpha^\delta \delta_\beta^\eta \Phi^{\rho\sigma} - \delta_\beta^\eta \delta_\alpha^\sigma \Phi^{\rho\delta} - \delta_\alpha^\eta \delta_\beta^\sigma \Phi^{\delta\rho} + 2\delta_\alpha^\rho \delta_\beta^\sigma g^{\gamma\delta} \partial_\gamma H^\eta \right. \\ & - g^{\delta\eta} \frac{\partial \Phi_{\alpha\beta}}{\partial g_{(\rho\sigma)}} + \delta_\beta^\delta g^{\rho\sigma} \partial_\alpha H^\eta + \delta_\alpha^\delta g^{\rho\sigma} \partial_\beta H^\eta - \delta_\alpha^\delta \delta_\beta^\eta g^{\mu\nu} \frac{\partial \Phi_{\mu\nu}}{\partial g_{(\rho\sigma)}} \\ & - \delta_\beta^\delta \delta_\alpha^\sigma g^{\rho\gamma} \partial_\gamma H^\eta - \delta_\alpha^\sigma g^{\rho\delta} \partial_\beta H^\eta + \delta_\beta^\eta g^{\gamma\delta} \frac{\partial \Phi_{\gamma\alpha}}{\partial g_{(\rho\sigma)}} \\ & \left. - \delta_\alpha^\delta \delta_\beta^\sigma g^{\rho\gamma} \delta_\gamma H^\eta - \delta_\beta^\sigma g^{\rho\delta} \partial_\alpha H^\eta + \delta_\alpha^\eta g^{\gamma\delta} \frac{\partial \Phi_{\gamma\beta}}{\partial g_{(\rho\sigma)}} \right\}. \end{aligned} \quad (36)$$

There are no $\partial_{\hat{\sigma}} \partial_\eta g_{\hat{\sigma}\hat{\sigma}}$ terms in the Einstein equations. The coefficients of these terms in (36) must therefore vanish. To extract these coefficients, we must first symmetrize the expression in curly brackets in δ and η (since $\partial_\delta \partial_\eta = \partial_\eta \partial_\delta$). With some cancellations, we get $\forall \alpha, \beta, \eta, \hat{\sigma}$

$$\begin{aligned} 0 = & \delta_\alpha^{\hat{\sigma}} \left\{ -g^{\hat{\sigma}\gamma} \delta_\beta^\eta \partial_\gamma H^{\hat{\sigma}} - g^{\hat{\sigma}\eta} \partial_\beta H^{\hat{\sigma}} - \delta_\beta^\eta g^{\mu\nu} \frac{\partial \Phi_{\mu\nu}}{\partial g_{\hat{\sigma}\hat{\sigma}}} + g^{\gamma\eta} \frac{\partial \Phi_{\gamma\beta}}{\partial g_{\hat{\sigma}\hat{\sigma}}} \right\} \\ & + \delta_\beta^{\hat{\sigma}} \left\{ -g^{\hat{\sigma}\gamma} \delta_\alpha^\eta \partial_\gamma H^{\hat{\sigma}} - g^{\hat{\sigma}\eta} \partial_\alpha H^{\hat{\sigma}} - \delta_\alpha^\eta g^{\mu\nu} \frac{\partial \Phi_{\mu\nu}}{\partial g_{\hat{\sigma}\hat{\sigma}}} + g^{\gamma\eta} \frac{\partial \Phi_{\gamma\alpha}}{\partial g_{\hat{\sigma}\hat{\sigma}}} \right\} \\ & + 2\delta_\alpha^{\hat{\sigma}} \delta_\beta^{\hat{\sigma}} g^{\gamma\eta} \partial_\gamma H^{\hat{\sigma}} - 2g^{\hat{\sigma}\eta} \frac{\partial \Phi_{\alpha\beta}}{\partial g_{\hat{\sigma}\hat{\sigma}}} \\ & + \delta_\alpha^\eta \left\{ g^{\hat{\sigma}\hat{\sigma}} \partial_\beta h^{\hat{\sigma}} + g^{\gamma\hat{\sigma}} \frac{\partial \Phi_{\gamma\beta}}{\partial g_{\hat{\sigma}\hat{\sigma}}} \right\} + \delta_\beta^\eta \left\{ g^{\hat{\sigma}\hat{\sigma}} \partial_\alpha h^{\hat{\sigma}} + g^{\gamma\hat{\sigma}} \frac{\partial \Phi_{\gamma\alpha}}{\partial g_{\hat{\sigma}\hat{\sigma}}} \right\}. \end{aligned} \quad (37)$$

In Eq. (37), we set $\alpha \neq \hat{\sigma} \neq \beta$ and $\alpha \neq \eta \neq \beta$. We obtain $\forall \alpha \neq \hat{\sigma} \neq \beta$

$$0 = \frac{\partial \Phi_{\alpha\beta}}{\partial g_{\hat{\sigma}\hat{\sigma}}}. \quad (38)$$

Next, we set $\alpha = \beta = \eta \neq \hat{\sigma}$. We get $\forall \hat{\alpha} \neq \hat{\sigma}$

$$0 = g^{\hat{\sigma}\hat{\sigma}} \partial_{\hat{\alpha}} H^{\hat{\sigma}} + g^{\gamma\hat{\sigma}} \frac{\partial \Phi_{\gamma\hat{\alpha}}}{\partial g_{\hat{\sigma}\hat{\sigma}}} - g^{\hat{\sigma}\hat{\alpha}} \frac{\partial \Phi_{\hat{\alpha}\hat{\alpha}}}{\partial g_{\hat{\sigma}\hat{\sigma}}}. \quad (39)$$

Owing to Eq. (38), this implies that $\forall \alpha \neq \hat{\sigma}$

$$0 = \partial_\alpha H^{\hat{\sigma}} + \frac{\partial \Phi_{\hat{\sigma}\alpha}}{\partial g_{\hat{\sigma}\hat{\sigma}}}. \quad (40)$$

Eqs. (38) and (40) are sufficient for (37) to vanish identically. This can be shown by considering in turn all remaining cases, namely (i) $\alpha \neq \hat{\sigma} \neq \beta$, $\alpha = \eta \neq \beta$; (ii) $\alpha \neq \hat{\sigma} \neq \beta$, $\alpha \neq \eta = \beta$; (iii) $\alpha = \hat{\sigma} \neq \beta$, $\alpha \neq \eta \neq \beta$; (iv) $\alpha \neq \hat{\sigma} = \beta$, $\alpha \neq \eta \neq \beta$; (v) $\alpha = \hat{\sigma} = \beta \neq \eta$; (vi) $\alpha \neq \hat{\sigma} = \beta$, $\alpha \neq \eta = \beta$; (vii) $\alpha = \hat{\sigma} \neq \beta$, $\alpha = \eta \neq \beta$; (viii) $\alpha \neq \hat{\sigma} = \beta$, $\alpha = \eta \neq \beta$; (ix) $\alpha = \hat{\sigma} \neq \beta$, $\alpha \neq \eta = \beta$; and (x) $\alpha = \hat{\sigma} = \eta = \beta$.

We now go back to Eq. (36), and consider terms of the form $\partial_{\hat{\sigma}} \partial_{\hat{\sigma}} g_{\rho\hat{\sigma}}$, for $\rho \neq \hat{\sigma}$. There are no such terms in the Einstein equations. Therefore, their coefficients must vanish. We get $\forall \alpha, \beta, \rho \neq \hat{\sigma}$

$$0 = -\delta_{\alpha}^{\hat{\sigma}} \delta_{\beta}^{\hat{\sigma}} \left\{ 2g^{\rho\gamma} \partial_{\gamma} H^{\hat{\sigma}} + g^{\mu\nu} \frac{\partial \Phi_{\mu\nu}}{\partial g_{\rho\hat{\sigma}}} \right\} + \delta_{\alpha}^{\hat{\sigma}} \left\{ \delta_{\beta}^{\rho} g^{\gamma\hat{\sigma}} \partial_{\gamma} H^{\hat{\sigma}} + g^{\rho\hat{\sigma}} \partial_{\beta} H^{\hat{\sigma}} + g^{\gamma\hat{\sigma}} \frac{\partial \Phi_{\gamma\beta}}{\partial g_{\rho\hat{\sigma}}} \right\} \\ + \delta_{\beta}^{\hat{\sigma}} \left\{ \delta_{\alpha}^{\rho} g^{\gamma\hat{\sigma}} \partial_{\gamma} H^{\hat{\sigma}} + g^{\rho\hat{\sigma}} \partial_{\alpha} H^{\hat{\sigma}} + g^{\gamma\hat{\sigma}} \frac{\partial \Phi_{\gamma\alpha}}{\partial g_{\rho\hat{\sigma}}} \right\} - g^{\hat{\sigma}\hat{\sigma}} \left\{ \delta_{\alpha}^{\rho} \partial_{\beta} H^{\hat{\sigma}} + \delta_{\beta}^{\rho} \partial_{\alpha} H^{\hat{\sigma}} + \frac{\partial \Phi_{\alpha\beta}}{\partial g_{\rho\hat{\sigma}}} \right\}. \quad (41)$$

Let us first consider the case where $\alpha \neq \hat{\sigma} \neq \beta$. We get (for α, β , and ρ all different from $\hat{\sigma}$)

$$0 = -g^{\hat{\sigma}\hat{\sigma}} \left\{ \delta_{\alpha}^{\rho} \partial_{\beta} H^{\hat{\sigma}} + \delta_{\beta}^{\rho} \partial_{\alpha} H^{\hat{\sigma}} + \frac{\partial \Phi_{\alpha\beta}}{\partial g_{\rho\hat{\sigma}}} \right\}. \quad (42)$$

Setting $\rho = \beta \neq \alpha$ in Eq. (42), we get $\forall \alpha, \hat{\rho}, \sigma \neq$

$$0 = \partial_{\alpha} H^{\sigma} + \frac{\partial \Phi_{\alpha\hat{\rho}}}{\partial g_{\hat{\rho}\sigma}}. \quad (43)$$

The case where $\rho = \alpha \neq \beta$ gives a similar result. Next, setting $\rho = \alpha = \beta$ yields $\forall \hat{\rho} \neq \sigma$

$$0 = 2\partial_{\hat{\rho}} H^{\sigma} + \frac{\partial \Phi_{\hat{\rho}\hat{\rho}}}{\partial g_{\hat{\rho}\sigma}}. \quad (44)$$

Finally, we have $\forall \alpha, \beta, \rho, \sigma$ such that $\alpha \neq \sigma \neq \beta$, $\alpha \neq \rho \neq \beta$, and $\rho \neq \sigma$

$$0 = \frac{\partial \Phi_{\alpha\beta}}{\partial g_{\rho\sigma}}. \quad (45)$$

Note that Eqs. (43) and (45) have no meaning in two dimensions.

Eqs. (43), (44), and (45) are sufficient for Eq. (41) to vanish identically. This can be shown by considering in turn all remaining cases, namely (i) $\alpha = \hat{\sigma} \neq \beta$; (ii) $\alpha \neq \hat{\sigma} = \beta$; and (iii) $\alpha = \hat{\sigma} = \beta$.

At this point, it is very useful to define a function $\tilde{\Phi}_{\alpha\beta}$ so that

$$\tilde{\Phi}_{\alpha\beta} = \Phi_{\alpha\beta} + g_{\alpha\gamma} \partial_{\beta} H^{\gamma} + g_{\gamma\beta} \partial_{\alpha} H^{\gamma}. \quad (46)$$

Clearly, $\tilde{\Phi}_{\alpha\beta} = \tilde{\Phi}_{\beta\alpha}$. Owing to (32), Eqs. (38), (40), (43), (44), and (45) imply

$$\frac{\partial \tilde{\Phi}_{\alpha\beta}}{\partial g_{\hat{\sigma}\hat{\sigma}}} = 0 \quad \text{if} \quad \alpha \neq \hat{\sigma} \neq \beta; \quad (47)$$

$$\frac{\partial \tilde{\Phi}_{\hat{\sigma}\beta}}{\partial g_{\hat{\sigma}\hat{\sigma}}} = 0 \quad \text{if} \quad \hat{\sigma} \neq \beta; \quad (48)$$

$$\frac{\partial \tilde{\Phi}_{\alpha\hat{\rho}}}{\partial g_{\hat{\rho}\sigma}} = 0 \quad \text{if} \quad \alpha, \hat{\rho}, \sigma \neq; \quad (49)$$

$$\frac{\partial \tilde{\Phi}_{\hat{\rho}\hat{\rho}}}{\partial g_{\hat{\rho}\sigma}} = 0 \quad \text{if} \quad \hat{\rho} \neq \sigma; \quad (50)$$

$$\frac{\partial \tilde{\Phi}_{\alpha\beta}}{\partial g_{\rho\sigma}} = 0 \quad \text{if} \quad \begin{cases} \alpha \neq \sigma \neq \beta, \\ \alpha \neq \rho \neq \beta, \\ \rho \neq \sigma. \end{cases} \quad (51)$$

Eqs. (47)–(51) mean that $\tilde{\Phi}_{\alpha\beta}$ is a function of $g_{\alpha\beta}$ (same indices) and x^λ alone.

The conditions we have obtained so far are necessary and sufficient for the coefficients of $\partial_{\hat{\sigma}}\partial_\eta g_{\hat{\sigma}\hat{\sigma}}$ and $\partial_{\hat{\sigma}}\partial_{\hat{\sigma}} g_{\hat{\sigma}\hat{\sigma}}$ terms to vanish. They do not, however, make all coefficients of $\partial\partial g$ terms in Eq. (36) equal to zero. And this is as it should be since, owing to Eq. (3), not all second derivatives of the metric tensor are independent.

∂g terms

There are many ∂g terms in Eq. (21). They come from (25) and (26), and from the fact that $\Gamma_{\tau\gamma\alpha}$ is related to $\partial_\alpha g_{\tau\gamma}$ through (6). Regrouping all terms and making use of (7), we get

$$\begin{aligned} \partial g \rightarrow & \frac{1}{2}g^{\gamma\delta}g^{\tau\rho} \{ [\partial_\alpha\Phi_{\tau\gamma} + \partial_\gamma\Phi_{\tau\alpha} - \partial_\tau\Phi_{\gamma\alpha}] \Gamma_{\rho\delta\beta} + [\partial_\beta\Phi_{\rho\delta} + \partial_\delta\Phi_{\rho\beta} - \partial_\rho\Phi_{\delta\beta}] \Gamma_{\tau\gamma\delta} \\ & - [\partial_\delta\Phi_{\tau\gamma} + \partial_\gamma\Phi_{\tau\delta} - \partial_\tau\Phi_{\gamma\delta}] \Gamma_{\rho\alpha\beta} - [\partial_\beta\Phi_{\rho\alpha} + \partial_\alpha\Phi_{\rho\beta} - \partial_\rho\Phi_{\alpha\beta}] \Gamma_{\tau\gamma\delta} \} \\ & + \frac{1}{2}g^{\gamma\delta} \left\{ (\partial_\gamma\partial_\delta H^\eta)(\Gamma_{\alpha\beta\eta} + \Gamma_{\beta\alpha\eta}) + (\partial_\alpha\partial_\beta H^\eta)(\Gamma_{\gamma\delta\eta} + \Gamma_{\delta\gamma\eta}) \right. \\ & - (\partial_\delta\partial_\beta H^\eta)(\Gamma_{\gamma\alpha\eta} + \Gamma_{\alpha\gamma\eta}) - (\partial_\alpha\partial_\gamma H^\eta)(\Gamma_{\beta\delta\eta} + \Gamma_{\delta\beta\eta}) \\ & - 2\partial_\gamma \left(\frac{\partial \tilde{\Phi}_{\alpha\beta}}{\partial g_{(\pi\sigma)}} \right) (\Gamma_{\pi\sigma\delta} + \Gamma_{\sigma\pi\delta}) \\ & - \partial_\alpha \left(\frac{\partial \tilde{\Phi}_{\gamma\delta}}{\partial g_{(\pi\sigma)}} \right) (\Gamma_{\pi\sigma\beta} + \Gamma_{\sigma\pi\beta}) - \partial_\beta \left(\frac{\partial \tilde{\Phi}_{\gamma\delta}}{\partial g_{(\pi\sigma)}} \right) (\Gamma_{\pi\sigma\alpha} + \Gamma_{\sigma\pi\alpha}) \\ & + \partial_\delta \left(\frac{\partial \tilde{\Phi}_{\gamma\alpha}}{\partial g_{(\pi\sigma)}} \right) (\Gamma_{\pi\sigma\beta} + \Gamma_{\sigma\pi\beta}) + \partial_\beta \left(\frac{\partial \tilde{\Phi}_{\gamma\alpha}}{\partial g_{(\pi\sigma)}} \right) (\Gamma_{\pi\sigma\delta} + \Gamma_{\sigma\pi\delta}) \\ & \left. + \partial_\gamma \left(\frac{\partial \tilde{\Phi}_{\delta\beta}}{\partial g_{(\pi\sigma)}} \right) (\Gamma_{\pi\sigma\alpha} + \Gamma_{\sigma\pi\alpha}) + \partial_\alpha \left(\frac{\partial \tilde{\Phi}_{\delta\beta}}{\partial g_{(\pi\sigma)}} \right) (\Gamma_{\pi\sigma\gamma} + \Gamma_{\sigma\pi\gamma}) \right\}. \end{aligned} \quad (52)$$

Let us substitute $\Phi_{\alpha\beta}$, as given in Eq. (46), in Eq. (52). Since $\tilde{\Phi}_{\alpha\beta}$ is a function of $g_{\alpha\beta}$ and x^λ only, we can write

$$\begin{aligned} \partial_\gamma \left(\frac{\partial \tilde{\Phi}_{\hat{\alpha}\hat{\beta}}}{\partial g_{(\pi\sigma)}} \right) (\Gamma_{\pi\sigma\delta} + \Gamma_{\sigma\pi\delta}) &= \partial_\gamma \left(\frac{\partial \tilde{\Phi}_{\hat{\alpha}\hat{\beta}}}{\partial g_{\hat{\alpha}\hat{\beta}}} X_{\hat{\alpha}\hat{\beta}}^{\pi\sigma} \right) (\Gamma_{(\pi\sigma)\delta} + \Gamma_{(\sigma\pi)\delta}) \\ &= \partial_\gamma \left(\frac{\partial \tilde{\Phi}_{\hat{\alpha}\hat{\beta}}}{\partial g_{\hat{\alpha}\hat{\beta}}} \right) (\Gamma_{\hat{\alpha}\hat{\beta}\delta} + \Gamma_{\hat{\beta}\hat{\alpha}\delta}). \end{aligned} \quad (53)$$

After cancellations and rearrangements, (52) becomes

$$\begin{aligned}
\partial g \rightarrow & \frac{1}{2} g^{\gamma\delta} g^{\tau\rho} \left\{ \left[\partial_{\hat{\alpha}} \tilde{\Phi}_{\tau\gamma} + \partial_{\gamma} \tilde{\Phi}_{\tau\hat{\alpha}} - \partial_{\tau} \tilde{\Phi}_{\gamma\hat{\alpha}} \right] \Gamma_{\rho\delta\hat{\beta}} + \left[\partial_{\hat{\beta}} \tilde{\Phi}_{\rho\delta} + \partial_{\delta} \tilde{\Phi}_{\rho\hat{\beta}} - \partial_{\rho} \tilde{\Phi}_{\delta\hat{\beta}} \right] \Gamma_{\tau\gamma\hat{\alpha}} \right. \\
& - \left[\partial_{\delta} \tilde{\Phi}_{\tau\gamma} + \partial_{\gamma} \tilde{\Phi}_{\tau\delta} - \partial_{\tau} \tilde{\Phi}_{\gamma\delta} \right] \Gamma_{\rho\hat{\alpha}\hat{\beta}} - \left[\partial_{\hat{\beta}} \tilde{\Phi}_{\rho\hat{\alpha}} + \partial_{\hat{\alpha}} \tilde{\Phi}_{\rho\hat{\beta}} - \partial_{\rho} \tilde{\Phi}_{\hat{\alpha}\hat{\beta}} \right] \Gamma_{\tau\gamma\delta} \Big\} \\
& + \frac{1}{2} g^{\gamma\delta} \left\{ -2\partial_{\gamma} \left(\frac{\partial \tilde{\Phi}_{\hat{\alpha}\hat{\beta}}}{\partial g_{\hat{\alpha}\hat{\beta}}} \right) (\Gamma_{\hat{\alpha}\hat{\beta}\delta} + \Gamma_{\hat{\beta}\hat{\alpha}\delta}) \right. \\
& - \partial_{\hat{\alpha}} \left(\frac{\partial \tilde{\Phi}_{\gamma\delta}}{\partial g_{\gamma\delta}} \right) (\Gamma_{\gamma\delta\hat{\beta}} + \Gamma_{\delta\gamma\hat{\beta}}) - \partial_{\hat{\beta}} \left(\frac{\partial \tilde{\Phi}_{\gamma\delta}}{\partial g_{\gamma\delta}} \right) (\Gamma_{\gamma\delta\hat{\alpha}} + \Gamma_{\delta\gamma\hat{\alpha}}) \\
& + \partial_{\delta} \left(\frac{\partial \tilde{\Phi}_{\gamma\hat{\alpha}}}{\partial g_{\gamma\hat{\alpha}}} \right) (\Gamma_{\gamma\hat{\alpha}\hat{\beta}} + \Gamma_{\hat{\alpha}\gamma\hat{\beta}}) + \partial_{\hat{\beta}} \left(\frac{\partial \tilde{\Phi}_{\gamma\hat{\alpha}}}{\partial g_{\gamma\hat{\alpha}}} \right) (\Gamma_{\gamma\hat{\alpha}\delta} + \Gamma_{\hat{\alpha}\gamma\delta}) \\
& \left. + \partial_{\gamma} \left(\frac{\partial \tilde{\Phi}_{\delta\hat{\beta}}}{\partial g_{\delta\hat{\beta}}} \right) (\Gamma_{\delta\hat{\beta}\hat{\alpha}} + \Gamma_{\hat{\beta}\delta\hat{\alpha}}) + \partial_{\hat{\alpha}} \left(\frac{\partial \tilde{\Phi}_{\delta\hat{\beta}}}{\partial g_{\delta\hat{\beta}}} \right) (\Gamma_{\delta\hat{\beta}\gamma} + \Gamma_{\hat{\beta}\delta\gamma}) \right\}. \tag{54}
\end{aligned}$$

There are no ∂g terms in the Einstein equations. Their coefficients must therefore vanish. Since the transformation $\partial_{\alpha} g_{\tau\gamma} \rightarrow \Gamma_{\tau\gamma\alpha}$ is nonsingular, the coefficients of Christoffel symbols must vanish. So we isolate $\Gamma_{\lambda\mu\nu}$ in (54), and set to zero its coefficient, symmetrized in μ and ν (since $\Gamma_{\lambda\mu\nu} = \Gamma_{\lambda\nu\mu}$). The result is $\forall \hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\mu}, \hat{\nu}$

$$\begin{aligned}
0 = & \left(\delta_{\hat{\alpha}}^{\hat{\mu}} \delta_{\hat{\beta}}^{\hat{\nu}} + \delta_{\hat{\alpha}}^{\hat{\nu}} \delta_{\hat{\beta}}^{\hat{\mu}} \right) \left\{ -g^{\gamma\delta} g^{\tau\hat{\lambda}} \left(\partial_{\delta} \tilde{\Phi}_{\tau\gamma} + \partial_{\gamma} \tilde{\Phi}_{\tau\delta} - \partial_{\tau} \tilde{\Phi}_{\gamma\delta} \right) \right. \\
& + g^{\hat{\lambda}\gamma} \partial_{\gamma} \left(\frac{\partial \tilde{\Phi}_{\hat{\lambda}\hat{\alpha}}}{\partial g_{\hat{\lambda}\hat{\alpha}}} \right) + g^{\hat{\lambda}\gamma} \partial_{\gamma} \left(\frac{\partial \tilde{\Phi}_{\hat{\lambda}\hat{\beta}}}{\partial g_{\hat{\lambda}\hat{\beta}}} \right) \Big\} \\
& + \left(\delta_{\hat{\alpha}}^{\hat{\lambda}} \delta_{\hat{\beta}}^{\hat{\mu}} + \delta_{\hat{\alpha}}^{\hat{\mu}} \delta_{\hat{\beta}}^{\hat{\lambda}} \right) \left\{ -2g^{\gamma\hat{\nu}} \partial_{\gamma} \left(\frac{\partial \tilde{\Phi}_{\hat{\alpha}\hat{\beta}}}{\partial g_{\hat{\alpha}\hat{\beta}}} \right) + g^{\gamma\hat{\nu}} \partial_{\gamma} \left(\frac{\partial \tilde{\Phi}_{\hat{\nu}\hat{\lambda}}}{\partial g_{\hat{\nu}\hat{\lambda}}} \right) \right\} \\
& + \left(\delta_{\hat{\alpha}}^{\hat{\lambda}} \delta_{\hat{\beta}}^{\hat{\nu}} + \delta_{\hat{\alpha}}^{\hat{\nu}} \delta_{\hat{\beta}}^{\hat{\lambda}} \right) \left\{ -2g^{\gamma\hat{\mu}} \partial_{\gamma} \left(\frac{\partial \tilde{\Phi}_{\hat{\alpha}\hat{\beta}}}{\partial g_{\hat{\alpha}\hat{\beta}}} \right) + g^{\gamma\hat{\mu}} \partial_{\gamma} \left(\frac{\partial \tilde{\Phi}_{\hat{\mu}\hat{\lambda}}}{\partial g_{\hat{\mu}\hat{\lambda}}} \right) \right\} \\
& + \delta_{\hat{\alpha}}^{\hat{\mu}} \left\{ g^{\hat{\nu}\delta} g^{\hat{\lambda}\rho} \left(\partial_{\hat{\beta}} \tilde{\Phi}_{\rho\delta} + \partial_{\delta} \tilde{\Phi}_{\rho\hat{\beta}} - \partial_{\rho} \tilde{\Phi}_{\delta\hat{\beta}} \right) - 2g^{\hat{\lambda}\hat{\nu}} \partial_{\hat{\beta}} \left(\frac{\partial \tilde{\Phi}_{\hat{\lambda}\hat{\nu}}}{\partial g_{\hat{\lambda}\hat{\nu}}} \right) + g^{\hat{\lambda}\hat{\nu}} \partial_{\hat{\beta}} \left(\frac{\partial \tilde{\Phi}_{\hat{\lambda}\hat{\alpha}}}{\partial g_{\hat{\lambda}\hat{\alpha}}} \right) \right\} \\
& + \delta_{\hat{\alpha}}^{\hat{\nu}} \left\{ g^{\hat{\mu}\delta} g^{\hat{\lambda}\rho} \left(\partial_{\hat{\beta}} \tilde{\Phi}_{\rho\delta} + \partial_{\delta} \tilde{\Phi}_{\rho\hat{\beta}} - \partial_{\rho} \tilde{\Phi}_{\delta\hat{\beta}} \right) - 2g^{\hat{\lambda}\hat{\mu}} \partial_{\hat{\beta}} \left(\frac{\partial \tilde{\Phi}_{\hat{\lambda}\hat{\mu}}}{\partial g_{\hat{\lambda}\hat{\mu}}} \right) + g^{\hat{\lambda}\hat{\mu}} \partial_{\hat{\beta}} \left(\frac{\partial \tilde{\Phi}_{\hat{\lambda}\hat{\alpha}}}{\partial g_{\hat{\lambda}\hat{\alpha}}} \right) \right\} \\
& + \delta_{\hat{\beta}}^{\hat{\mu}} \left\{ g^{\gamma\hat{\nu}} g^{\tau\hat{\lambda}} \left(\partial_{\hat{\alpha}} \tilde{\Phi}_{\tau\gamma} + \partial_{\gamma} \tilde{\Phi}_{\tau\hat{\alpha}} - \partial_{\tau} \tilde{\Phi}_{\gamma\hat{\alpha}} \right) - 2g^{\hat{\lambda}\hat{\nu}} \partial_{\hat{\alpha}} \left(\frac{\partial \tilde{\Phi}_{\hat{\lambda}\hat{\nu}}}{\partial g_{\hat{\lambda}\hat{\nu}}} \right) + g^{\hat{\lambda}\hat{\nu}} \partial_{\hat{\alpha}} \left(\frac{\partial \tilde{\Phi}_{\hat{\lambda}\hat{\beta}}}{\partial g_{\hat{\lambda}\hat{\beta}}} \right) \right\} \\
& + \delta_{\hat{\beta}}^{\hat{\nu}} \left\{ g^{\gamma\hat{\mu}} g^{\tau\hat{\lambda}} \left(\partial_{\hat{\alpha}} \tilde{\Phi}_{\tau\gamma} + \partial_{\gamma} \tilde{\Phi}_{\tau\hat{\alpha}} - \partial_{\tau} \tilde{\Phi}_{\gamma\hat{\alpha}} \right) - 2g^{\hat{\lambda}\hat{\mu}} \partial_{\hat{\alpha}} \left(\frac{\partial \tilde{\Phi}_{\hat{\lambda}\hat{\mu}}}{\partial g_{\hat{\lambda}\hat{\mu}}} \right) + g^{\hat{\lambda}\hat{\mu}} \partial_{\hat{\alpha}} \left(\frac{\partial \tilde{\Phi}_{\hat{\lambda}\hat{\beta}}}{\partial g_{\hat{\lambda}\hat{\beta}}} \right) \right\} \\
& + \delta_{\hat{\alpha}}^{\hat{\lambda}} g^{\hat{\mu}\hat{\nu}} \left\{ \partial_{\hat{\beta}} \left(\frac{\partial \tilde{\Phi}_{\hat{\mu}\hat{\alpha}}}{\partial g_{\hat{\mu}\hat{\alpha}}} \right) + \partial_{\hat{\beta}} \left(\frac{\partial \tilde{\Phi}_{\hat{\nu}\hat{\alpha}}}{\partial g_{\hat{\nu}\hat{\alpha}}} \right) \right\} + \delta_{\hat{\beta}}^{\hat{\lambda}} g^{\hat{\mu}\hat{\nu}} \left\{ \partial_{\hat{\alpha}} \left(\frac{\partial \tilde{\Phi}_{\hat{\mu}\hat{\beta}}}{\partial g_{\hat{\mu}\hat{\beta}}} \right) + \partial_{\hat{\alpha}} \left(\frac{\partial \tilde{\Phi}_{\hat{\nu}\hat{\beta}}}{\partial g_{\hat{\nu}\hat{\beta}}} \right) \right\} \\
& - 2g^{\hat{\mu}\hat{\nu}} g^{\hat{\lambda}\rho} \left(\partial_{\hat{\beta}} \tilde{\Phi}_{\rho\hat{\alpha}} + \partial_{\hat{\alpha}} \tilde{\Phi}_{\rho\hat{\beta}} - \partial_{\rho} \tilde{\Phi}_{\hat{\alpha}\hat{\beta}} \right). \tag{55}
\end{aligned}$$

In Eq. (55), we let $\hat{\alpha} \neq \hat{\lambda} \neq \hat{\beta}$, $\hat{\alpha} \neq \hat{\mu} \neq \hat{\beta}$, and $\hat{\alpha} \neq \hat{\nu} \neq \hat{\beta}$. We obtain, $\forall \alpha, \beta$ and $\forall \lambda$ such that $\alpha \neq \lambda \neq \beta$

$$0 = g^{\lambda\rho} \left\{ \partial_\beta \tilde{\Phi}_{\rho\alpha} + \partial_\alpha \tilde{\Phi}_{\rho\beta} - \partial_\rho \tilde{\Phi}_{\alpha\beta} \right\}. \quad (56)$$

If the value of λ was not restricted, we would easily conclude that $\partial_\beta \tilde{\Phi}_{\kappa\alpha}$ vanished $\forall \beta, \kappa, \alpha$. The conclusion probably holds also with the restrictions on λ . Indeed it is unlikely that Eq. (56) holds identically with nonvanishing values of $\partial_\beta \tilde{\Phi}_{\kappa\alpha}$, owing to the fact that $\tilde{\Phi}_{\kappa\alpha}$ does not involve metric components other than $g_{\kappa\alpha}$. If $\partial_\beta \tilde{\Phi}_{\kappa\alpha} = 0$, one immediately concludes that Eq. (55) holds identically.

Thus, a calculation of $\tilde{\Phi}_{\kappa\alpha}$ could proceed as follows: (i) attempt to prove that $\partial_\beta \tilde{\Phi}_{\kappa\alpha} = 0$, with the result that $\tilde{\Phi}_{\hat{\kappa}\hat{\alpha}} = \tilde{\Phi}_{\hat{\kappa}\hat{\alpha}}(g_{\hat{\kappa}\hat{\alpha}})$; (ii) find the conditions on $\tilde{\Phi}_{\kappa\alpha}$ given by the vanishing of the independent $\partial\partial g$ terms (not already discussed), of the $\partial g\partial g$ terms and of the no-derivative terms (including contributions coming from the elimination, by Eq. (3), of the dependent $\partial\partial g$ terms). Such a calculation, especially step (ii), would be very complicated. Fortunately, there is a way around.

5 Lie algebra of the generators

The generator v of a Lie symmetry of Einstein's equations has the form of Eq. (18). We have shown, in the last section, that $H^\mu = H^\mu(x^\lambda)$ and that $\Phi_{\mu\nu}$ is given as in Eq. (46) with $\tilde{\Phi}_{\hat{\mu}\hat{\nu}} = \tilde{\Phi}_{\hat{\mu}\hat{\nu}}(x^\lambda, g_{\hat{\mu}\hat{\nu}})$. We can thus write

$$v = H^\mu(x^\lambda) \frac{\partial}{\partial x^\mu} + \left\{ -g_{\mu\gamma} \partial_\nu H^\gamma(x^\lambda) - g_{\gamma\nu} \partial_\mu H^\gamma(x^\lambda) + \tilde{\Phi}_{\mu\nu}(x^\lambda, g_{\mu\nu}) \right\} \frac{\partial}{\partial g_{(\mu\nu)}}. \quad (57)$$

Let L be the set of all v that generate Lie symmetries of Einstein's equations. Then L is a Lie algebra. It is well known that general coordinate transformations are symmetries of Einstein's equations. Such transformations are generated by

$$v^{GCT} = f^\alpha \frac{\partial}{\partial x^\alpha} - \{g_{\alpha\gamma} \partial_\beta f^\gamma + g_{\gamma\beta} \partial_\alpha f^\gamma\} \frac{\partial}{\partial g_{(\alpha\beta)}}, \quad (58)$$

where f^α is any function of x^λ . Thus, v^{GCT} belongs to L .

Let v , given as in Eq. (57), belong to L . Since L is a Lie algebra, $v - v^{GCT}$ also belongs to L . Setting $f^\alpha = H^\alpha$, we get that

$$\tilde{v} = \tilde{\Phi}_{\mu\nu}(x^\lambda, g_{\mu\nu}) \frac{\partial}{\partial g_{(\mu\nu)}}, \quad (59)$$

with $\tilde{\Phi}_{\mu\nu}$ given by Eq. (46), also belongs to L . Moreover, the commutator of \tilde{v} with any v^{GCT} belongs to L . This, we shall now show, severely limits the form of the function $\tilde{\Phi}_{\mu\nu}$.

It is not difficult to show that, for any $f^\alpha(x^\lambda)$ and any $\tilde{\Phi}_{\mu\nu}(x^\lambda, g_{\mu\nu})$, the commutator of \tilde{v} with v^{GCT} is given by

$$[\tilde{v}, v^{GCT}] = F_{\mu\nu} \frac{\partial}{\partial g_{(\mu\nu)}}, \quad (60)$$

where for $\mu \leq \nu$

$$F_{\mu\nu} = -f^\alpha \partial_\alpha \tilde{\Phi}_{\mu\nu} - \tilde{\Phi}_{\mu\gamma} \partial_\nu f^\gamma - \tilde{\Phi}_{\gamma\nu} \partial_\mu f^\gamma + (g_{\alpha\gamma} \partial_\beta f^\gamma + g_{\gamma\beta} \partial_\alpha h^\gamma) \frac{\partial \tilde{\Phi}_{\mu\nu}}{\partial g_{(\alpha\beta)}}. \quad (61)$$

Eq. (61) will hold for all μ and ν if we set $F_{\mu\nu} = F_{\nu\mu}$ when $\mu > \nu$. Since $\tilde{\Phi}_{\mu\nu}$ is a function of x^λ and $g_{\mu\nu}$ only, we have $\forall \hat{\mu}, \hat{\nu}$

$$F_{\hat{\mu}\hat{\nu}} = -f^\alpha \partial_\alpha \tilde{\Phi}_{\hat{\mu}\hat{\nu}} - \tilde{\Phi}_{\hat{\mu}\gamma} \partial_{\hat{\nu}} f^\gamma - \tilde{\Phi}_{\gamma\hat{\nu}} \partial_{\hat{\mu}} f^\gamma + (g_{\hat{\mu}\gamma} \partial_{\hat{\nu}} f^\gamma + g_{\gamma\hat{\nu}} \partial_{\hat{\mu}} h^\gamma) \frac{\partial \tilde{\Phi}_{\hat{\mu}\hat{\nu}}}{\partial g_{\hat{\mu}\hat{\nu}}}. \quad (62)$$

For the right-hand side of (60) to belong to L , it is necessary that $F_{\hat{\mu}\hat{\nu}}$ be a function of x^λ and $g_{\hat{\mu}\hat{\nu}}$ only. From Eq. (62), this implies that

$$\begin{aligned} & \left(-\tilde{\Phi}_{\hat{\mu}\gamma} + g_{\hat{\mu}\gamma} \frac{\partial \tilde{\Phi}_{\hat{\mu}\hat{\nu}}}{\partial g_{\hat{\mu}\hat{\nu}}} \right) \partial_{\hat{\nu}} f^\gamma + \left(-\tilde{\Phi}_{\gamma\hat{\nu}} + g_{\gamma\hat{\nu}} \frac{\partial \tilde{\Phi}_{\hat{\mu}\hat{\nu}}}{\partial g_{\hat{\mu}\hat{\nu}}} \right) \partial_{\hat{\mu}} f^\gamma \\ &= F_{\hat{\mu}\hat{\nu}}(x^\lambda, g_{\hat{\mu}\hat{\nu}}) + f^\alpha \partial_\alpha \tilde{\Phi}_{\hat{\mu}\hat{\nu}} \equiv \tilde{F}_{\hat{\mu}\hat{\nu}}(x^\lambda, g_{\hat{\mu}\hat{\nu}}). \end{aligned} \quad (63)$$

Since f^γ is arbitrary, we can set $f^\gamma(x^\lambda) = \delta_\rho^\gamma f(x^\sigma)$, for ρ and σ fixed. Letting \dot{f} denote the derivative of f with respect to its argument, we obtain $\forall \rho, \sigma, \hat{\mu}, \hat{\nu}$

$$\left(-\tilde{\Phi}_{\hat{\mu}\rho} + g_{\hat{\mu}\rho} \frac{\partial \tilde{\Phi}_{\hat{\mu}\hat{\nu}}}{\partial g_{\hat{\mu}\hat{\nu}}} \right) \delta_\nu^\sigma \dot{f} + \left(-\tilde{\Phi}_{\rho\hat{\nu}} + g_{\rho\hat{\nu}} \frac{\partial \tilde{\Phi}_{\hat{\mu}\hat{\nu}}}{\partial g_{\hat{\mu}\hat{\nu}}} \right) \delta_\mu^\sigma \dot{f} = \tilde{F}_{\hat{\mu}\hat{\nu}}(x^\lambda, g_{\hat{\mu}\hat{\nu}}). \quad (64)$$

Let $\hat{\mu} = \hat{\nu} = \sigma$. We have $\forall \hat{\mu}, \rho$

$$2 \left(-\tilde{\Phi}_{\hat{\mu}\rho} + g_{\hat{\mu}\rho} \frac{\partial \tilde{\Phi}_{\hat{\mu}\hat{\mu}}}{\partial g_{\hat{\mu}\hat{\mu}}} \right) \dot{f} = \tilde{F}_{\hat{\mu}\hat{\mu}}(x^\lambda, g_{\hat{\mu}\hat{\mu}}). \quad (65)$$

Redefining \tilde{F} as $\tilde{F}/2\dot{f}$, we get

$$-\tilde{\Phi}_{\hat{\mu}\rho} + g_{\hat{\mu}\rho} \frac{\partial \tilde{\Phi}_{\hat{\mu}\hat{\mu}}}{\partial g_{\hat{\mu}\hat{\mu}}} = \tilde{F}_{\hat{\mu}\hat{\mu}}(x^\lambda, g_{\hat{\mu}\hat{\mu}}). \quad (66)$$

Eq. (66) holds identically for $\rho = \hat{\mu}$. For $\hat{\rho} \neq \hat{\mu}$, it implies that

$$\tilde{\Phi}_{\hat{\mu}\hat{\rho}} = A_{\hat{\mu}\hat{\rho}} g_{\hat{\mu}\hat{\rho}} + B_{\hat{\mu}\hat{\rho}}, \quad (67)$$

where $A_{\hat{\mu}\hat{\rho}}$ and $B_{\hat{\mu}\hat{\rho}}$ are symmetric in $\hat{\mu}$ and $\hat{\rho}$ and are arbitrary functions of x^λ . Substituting (67) in (66), we find that $\forall \hat{\rho} \neq \hat{\mu}$

$$\frac{\partial \tilde{\Phi}_{\hat{\mu}\hat{\mu}}}{\partial g_{\hat{\mu}\hat{\mu}}} = A_{\hat{\mu}\hat{\mu}}, \quad (68)$$

or

$$\tilde{\Phi}_{\hat{\mu}\hat{\mu}} = A_{\hat{\mu}\hat{\mu}} g_{\hat{\mu}\hat{\mu}} + B_{\hat{\mu}\hat{\mu}}, \quad (69)$$

where $B_{\hat{\mu}\hat{\mu}}$ is an arbitrary function of x^λ . Eqs. (68) and (69) imply that, for $\hat{\rho} \neq \hat{\mu}$, $A_{\hat{\mu}\hat{\rho}}$ is independent of $\hat{\rho}$ and, since $A_{\hat{\mu}\hat{\rho}}$ is symmetric, independent of $\hat{\mu}$ also. Defining $A = A_{\hat{\mu}\hat{\rho}}$, we can write Eqs. (67) and (69) as

$$\tilde{\Phi}_{\mu\rho} = A(x^\lambda) g_{\mu\rho} + B_{\mu\rho}(x^\lambda), \quad (70)$$

which now holds $\forall \rho, \mu$. Eq. (70) is a necessary condition for the commutator of \tilde{v} and v^{GCT} to belong to L .

Let us now substitute (70) in (56). We obtain $\forall \alpha, \beta$ such that $\alpha \neq \lambda \neq \beta$

$$0 = -[\partial_\rho A]g^{\lambda\rho}g_{\alpha\beta} + g^{\lambda\rho}\{\partial_\beta B_{\rho\alpha} + \partial_\alpha B_{\rho\beta} - \partial_\rho B_{\alpha\beta}\}. \quad (71)$$

This must hold as an identity. Thus, terms made up of different powers of the metric components must separately vanish. That is,

$$0 = [\partial_\rho A]g^{\lambda\rho}g_{\alpha\beta} \quad (72)$$

and

$$0 = g^{\lambda\rho}\{\partial_\beta B_{\rho\alpha} + \partial_\alpha B_{\rho\beta} - \partial_\rho B_{\alpha\beta}\}. \quad (73)$$

From (72), and since the $g^{\lambda\rho}$ are independent variables, we get $\forall \rho$

$$0 = \partial_\rho A. \quad (74)$$

In three or more dimensions, Eq. (73) implies similarly that $\forall \alpha, \beta, \rho$

$$0 = \partial_\beta B_{\rho\alpha} + \partial_\alpha B_{\rho\beta} - \partial_\rho B_{\alpha\beta}. \quad (75)$$

This, in turn, implies that $\forall \alpha, \beta, \rho$

$$0 = \partial_\beta B_{\rho\alpha}. \quad (76)$$

Therefore, in three or more dimensions, A and $B_{\mu\rho}$ in Eq. (70) must be constant. That is,

$$\tilde{\Phi}_{\mu\rho} = Ag_{\mu\rho} + B_{\mu\rho}. \quad (77)$$

Now the Lie algebra property of L once more limits the form of $\tilde{\Phi}_{\mu\rho}$. Starting from Eq. (77), and going through an argument similar to the one between Eqs. (57) and (70), we find that the commutator (60) belongs to L only if $\forall \mu, \rho$, $B_{\mu\rho} = 0$.

In two dimensions, the situation is slightly more complicated. Eq. (74) still holds, but the constraint $\alpha \neq \lambda \neq \beta$ before Eq. (71) implies that $\alpha = \beta$. Eq. (75) then reads, $\forall \hat{\alpha}, \rho$,

$$0 = 2\partial_{\hat{\alpha}} B_{\rho\hat{\alpha}} - \partial_\rho B_{\hat{\alpha}\hat{\alpha}}. \quad (78)$$

Let $\hat{\alpha} = \rho$. We have $\partial_{\hat{\alpha}} B_{\hat{\alpha}\hat{\alpha}} = 0$, so that

$$B_{00} = B_{00}(x^1), \quad B_{11} = B_{11}(x^0). \quad (79)$$

With $\hat{\alpha} \neq \rho$, we have

$$2\partial_0 B_{10} = \partial_1 B_{00}, \quad 2\partial_1 B_{01} = \partial_0 B_{11}, \quad (80)$$

whence, from (79)

$$\partial_0 \partial_0 B_{10} = 0 = \partial_1 \partial_1 B_{10}. \quad (81)$$

This implies that

$$B_{10} = ax^0 x^1 + bx^0 + cx^1 + d, \quad (82)$$

where a , b , c , and d are constants. Substituting (82) in (80), we obtain

$$B_{00} = a(x^1)^2 + 2bx^1 + f, \quad B_{11} = a(x^0)^2 + 2cx^0 + g, \quad (83)$$

where f and g are constants.

The upshot is that, in two dimensions, Eq. (70) holds with A constant and $B_{\mu\rho}$ given as in Eqs. (82) and (83). But again, the Lie algebra property of L limits the form of $\tilde{\Phi}_{\mu\rho}$, and we can show that $\forall \mu, \rho$, $B_{\mu\rho} = 0$.

To sum up, we have shown that the generator of a Lie symmetry of Einstein's vacuum equations in N dimensions necessarily has the form of Eq. (57), with $H^\mu(x^\lambda)$ arbitrary and $\tilde{\Phi}_{\mu\nu} = Ag_{\mu\nu}$. The functions $H^\mu(x^\lambda)$ correspond to general coordinate transformations. The constant A , on the other hand, corresponds to uniform scale transformations of the metric. It is easy to check that such transformations leave Einstein's equations invariant if and only if the cosmological term vanishes. Provided the system (3) is nondegenerate [1], we have thus shown that all Lie symmetries of Einstein's vacuum equations in N dimensions are obtained from general coordinate transformations and, when the cosmological term vanishes, uniform rescalings of the metric.

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